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# Network synthesis for prescribed transient response using trigonometric series approximations.

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Massachusetts Institute of Technology

NETWORK SYNTHESIS FOR
PRESCRIBED TRANSIENT RESPONSE
USING TRIGONOMETRIC SERIES APPROXIMATIONS

by

WILLIAM BLOUNT RODMAN IV

Course VI June 18, 1952

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# NETWORK SYNTHESIS FOR PRESCRIBED TRANSIENT RESPONSE USING TRIGONOMETRIC SERIES APPROXIMATIONS

by

WILLIAM BLOUNT RODMAN IV

B.S., U.S. Naval Academy
(1941)

B.S., U.S. Naval Postgraduate School
(1951)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
(1952)

Signature of Author________________________________
Department of Electrical Engineering,
June 18, 1952.

Certified by________________________________
Thesis Supervisor

________________________________
Chairman, Departmental Committee on Graduate Students

ABSTRACT

NETWORK SYNTHESIS FOR PRESCRIBED TRANSIENT
RESPONSE USING TRIGONOMETRIC SERIES APPROXIMATIONS

by

WILLIAM BLOUNT RODMAN IV

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL ENGINEERING
ON 18 JUNE, 1952, IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF MASTER OF SCIENCE.

Dr. E. A. Guillemin has proposed a means of network synthesis for prescribed transient response which permits approximations to be made in the time domain rather than the frequency domain. The desired transient is obtained from an appropriate combination of auxiliary periodic functions such that the result cancels everywhere except over the period of the transient. The system function is obtained from the transforms of trigonometric series approximations to these auxiliary periodic functions. The system function thus obtained is not always realizable, but it appears that approximations can be made such that it will be realizable.

On the assumption that the system function be realizable, or that it can be made so, a synthesis procedure is developed. The transient is decomposed into two components having, respectively, even and odd symmetry about the midpoint. The system functions determined for these components satisfy the requirements for synthesis as a lossless network terminated in a resistance. Such networks are synthesized for each component. These two networks are then connected "back to back" to realize a network for the overall system function.

THESIS SUPERVISOR E. A. Guillemin

TITLE PROFESSOR OF ELECTRICAL ENGINEERING

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# INTRODUCTION

The problem of synthesizing a finite, lumped parameter, linear, passive network for prescribed transient response has been attacked by forming the ratio of the transforms of the output and input functions to give the system function h(s). The system function has then been synthesized into a network with the necessary approximations being made in the frequency domain. Results obtained have varied from good to poor, and it appears that the degree of approximation obtained in the time domain cannot readily be determined from the degree of approximation made in the frequency domain.

A means of making the required approximations in the time domain, instead of the frequency domain, would permit control of the time form of the transient. One such means available is the finite trigonometric series, which can be made to approximate a given periodic time function to any desired degree of accuracy. An appropriate combination of periodic functions based on the desired transient can then be made such that the combination cancels everywhere except over the period of the transient.

# INTRODUCTION

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# CHAPTER I

## DERIVATION OF THE TRANSFER FUNCTION TO BE REALIZED

The following method of representing a transient of finite duration by an appropriate combination of semi-periodic functions has been proposed.[1] Define the desired transient as f(t).

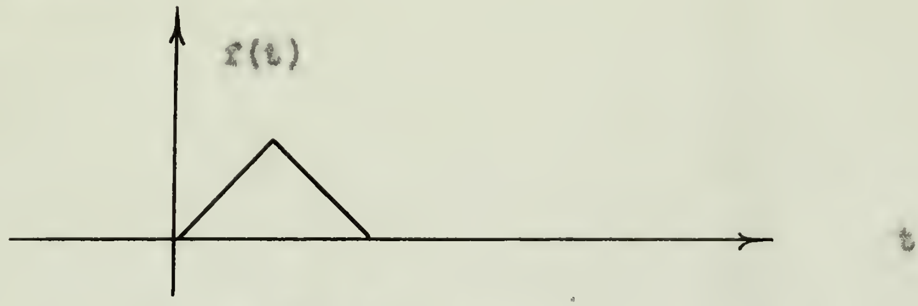



Fig. 1
The Desired Transient

Define a semi-periodic function, $f_p(t)$ as follows:

$$f_p(t) \equiv \begin{cases} 0 & t < 0 \\ f(t) & 0 < t < \frac{\tau}{2} \\ 0 & \frac{\tau}{2} < t < \tau \end{cases}$$

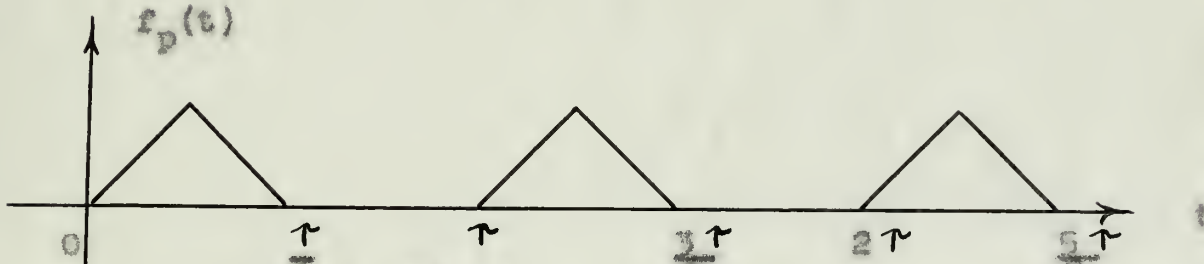



Fig. 2
Semi-Periodic Function $f_p(t)$ Derived from f(t)

# CHAPTER 2

## DERIVATION OF THE FOURIER TRANSFORM OF [illegible]

The following method of representing a transient of finite duration by an equivalent combination of semi-periodic functions has been proposed.[illegible] Define the desired transient as $f(t)$.

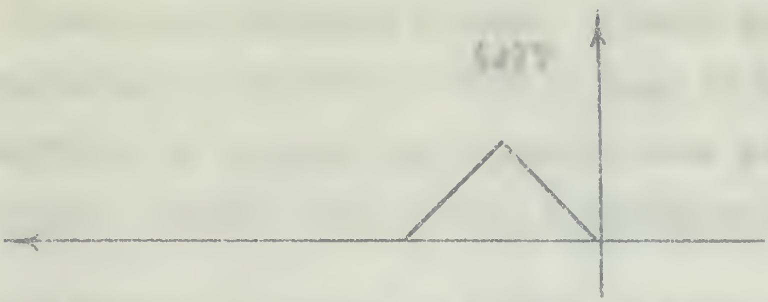



Fig. 1
The Desired Transient

Define a semi-periodic function, $f_s(t)$ as follows:

$$f_s(t) \equiv \begin{cases} 0 & t < 0 \\ f(t) & 0 < t < \frac{T}{2} \\ [illegible] & \frac{T}{2} < t < T \end{cases}$$


Figure 2
Semi-periodic function $f_s(t)$ derived from $f(t)$

$f_p(t)$ is representable in a Fourier series as:

$$f_p(t) = \sum_{k=-\infty}^{\infty} \alpha_k \epsilon^{jk\omega t} \qquad (t > 0)$$

and its transform $h_p(s)$ is represented as:

$$h_p(s) = \sum_{k=-\infty}^{\infty} \frac{\alpha_k}{s - jk\omega} \ .$$

Now define two additional functions, $f_1(t)$ and $f_2(t)$ as follows:

$$f_1(t) \equiv f_p(t) + f_p(t - \frac{\tau}{2})$$

$$f_2(t) \equiv f_p(t) - f_p(t - \frac{\tau}{2})$$

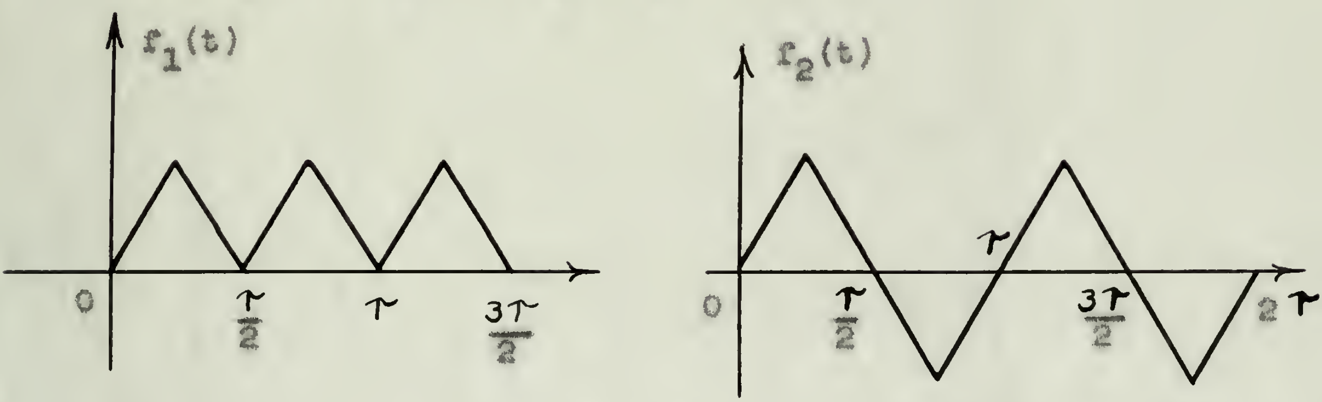



Fig. 3
Functions $f_1(t)$ and $f_2(t)$, Derived From $f_p(t)$

The function

$$f_p(t - \frac{\tau}{2}) = \sum_{k=-\infty}^{\infty} \alpha_k \epsilon^{jk\omega(t - \frac{\tau}{2})} = \sum_{k=-\infty}^{\infty} \alpha_k \epsilon^{jk\omega t} \cdot \epsilon^{-jk\omega\frac{\tau}{2}} \ .$$

$f_p(t)$ is representable in a Fourier series as:

$$f_p(t) = \sum_{k=-\infty}^{\infty} \alpha_k \varepsilon^{jk\omega t}, \qquad (t > 0)$$

and its transform $F_p(s)$ is calculated as:

$$F_p(s) = \sum_{k=-\infty}^{\infty} \frac{\alpha_k}{s - jk\omega} .$$

Now define two additional functions, $f_1(t)$ and $f_2(t)$ as follows:

$$f_1(t) \equiv f_p(t) + f_p(t - \frac{T}{2})$$

$$f_2(t) \equiv f_p(t) - f_p(t - \frac{T}{2})$$

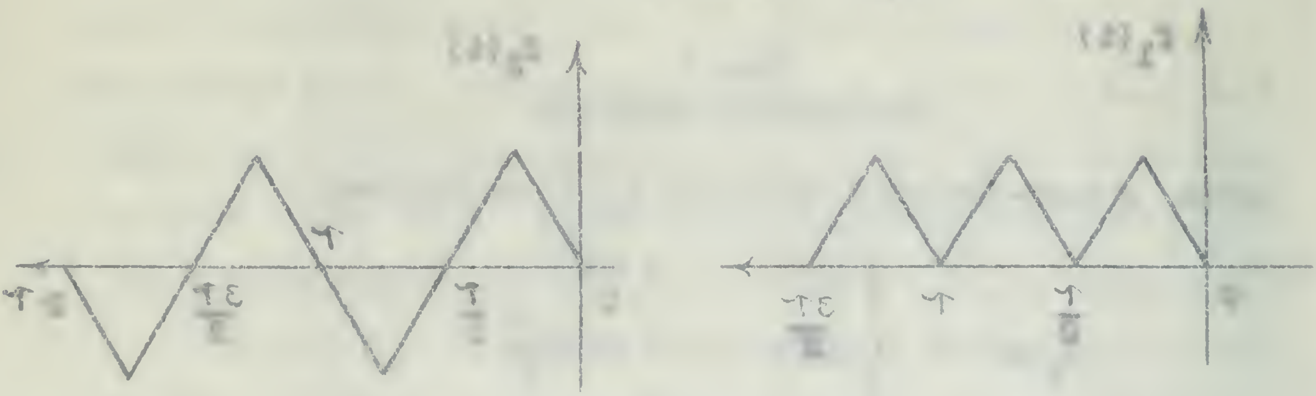



Fig. 3
Functions $f_1(t)$ and $f_2(t)$, Derived From $f_p(t)$

The function

$$f_p(t - \frac{T}{2}) = \sum_{k=-\infty}^{\infty} \alpha_k \varepsilon^{jk\omega(t - \frac{T}{2})} = \sum_{k=-\infty}^{\infty} \alpha_k \varepsilon^{jk\omega t} \varepsilon^{-jk\omega\frac{T}{2}} .$$

Note that $\frac{\omega\tau}{2} = \pi$

then $f_p(t-\frac{\tau}{2}) = \sum_{k=-\infty}^{\infty} \alpha_k \epsilon^{jk\omega t} \cdot \epsilon^{-jk\pi} = \sum_{k=0,\pm 2,4\cdots}^{\infty} \alpha_k \epsilon^{jk\omega t}$

$-\sum_{k=\pm 1,3,5\cdots}^{\infty} \alpha_k \epsilon^{jk\omega t}$

whence $f_1(t) = 2\sum_{k=0,\pm 2,4\cdots}^{\infty} \alpha_k \epsilon^{jk\omega t}$

and $\quad f_2(t) = 2\sum_{k=\pm 1,3,5\cdots}^{\infty} \alpha_k \,^{jk\omega t}$ .

The transforms of $f_1(t)$ and $f_2(t)$ are:

$$h_1(s) = h_p(s)\left[1+\epsilon^{-\frac{s\tau}{2}}\right] = 2\sum_{k=0,\pm 2,4\cdots}^{n} \frac{\alpha_k}{s-jk\omega}$$

$$h_2(s) = h_p(s)\left[1-\epsilon^{-\frac{s\tau}{2}}\right] = 2\sum_{k=\pm 1,3,5\cdots}^{\infty} \frac{\alpha_k}{s-jk\omega} \quad .$$

Following a line of physical reasoning, these functions are combined to form a new function, h(s) given by $h(s) \equiv \frac{h_1 h_2}{h_1+h_2}$. Direct substitution shows that $h(s) = \frac{h_p(s)}{2}\left[1-\epsilon^{-s\tau}\right]$. The inverse transform of h(s) is readily recognized as

$$\frac{1}{2}\left\{f_p(t)-f_p(t-\tau)\right\} = \frac{1}{2}\, f(t).$$

In general the Fourier series representation of $f_p(t)$ will be an infinite series. Representing finite approximations by $^*$, then:

$$f_p^*(t) = \sum_{k=-n}^{n} \alpha_k \epsilon^{jk\omega t} \qquad (t > 0)$$

$$h_p^*(s) = \sum_{k=-n}^{n} \frac{\alpha_k}{s - jk\omega} \equiv \frac{P(s)}{Q(s)}$$

$$h_1^*(s) = \sum_{k=0,\pm 2,4\cdots}^{n} \frac{2\alpha_k}{s - jk\omega} \equiv \frac{P_1(s)}{Q_1(s)}$$

$$h_2^*(s) = \sum_{k=\pm 1,3,5\cdots}^{n} \frac{2\alpha_k}{s - jk\omega} \equiv \frac{P_2(s)}{Q_2(s)}$$

$$h_1^*(s) + h_2^*(s) = 2h_p^*(s) = \frac{2P(s)}{Q(s)}$$

$$h_1^* \cdot h_2^* = \frac{P_1P_2}{Q_1Q_2} = \frac{P_1P_2}{Q}$$

$$h^*(s) = \frac{P_1P_2}{2P}$$

# CHAPTER II

## DEVELOPMENT OF A SYNTHESIS PROCEDURE

The purpose here is to develop a practical means of synthesis of a network whose system function is $h^*(s)$, that is, a network whose response to a unit impulse function is $f^*(t)$.

The necessary and sufficient conditions[2] for the realizability of a network whose transfer impedance# is $h^*(s)$ are:

(a) The numerator of $h^*(s)$ be of degree not greater than the denominator and,

(b) The denominator have no zeros in the right half s-plane, i.e., the denominator must be a Hurwitz polynomial.

That condition (a) is fulfilled is evident from an expansion of the functions involved. The fulfillment of condition (b) is subject to discussion beyond the scope of this work. It can be demonstrated that, using the Fourier coefficients in a finite approximation of $f_p(t)$, certain functions yield a denominator polynomial which is Hurwitz while others do not. There appears to be a possibility of so selecting the coefficients that the denominator polynomial will be Hurwitz. In any event, this condition must

# Note: Throughout the following development the word "admittance" may be substituted for "impedance" without invalidating the arguments.

be met before synthesis is possible.

Assume, therefore, that means have been devised to assure the Hurwitz character of the denominator polynomial, or that it is desired to synthesize a network for a function which yields one. It is advisable, in the interests of brevity, to normalize at $\omega = 1$, and to shift from complex coefficients to trigonometric coefficients.

Let the duration of $f^*(t)$ be $\pi$, then the period of $f_p^*(t)$ is $2\pi$, and $\omega = \frac{2\pi}{\tau} = 1$.

$$h_p^*(s) = \sum_{k=-n}^{n} \frac{\alpha_k}{s - jk} = \frac{\alpha_0}{s} + \sum_{k=1}^{n} \frac{(\alpha_k + \alpha_{-k})s + jk(\alpha_k - \alpha_{-k})}{s^2 + k^2}.$$

Noting that $(\alpha_k + \alpha_{-k})$ and $j(\alpha_k - \alpha_{-k})$ are respectively $a_k$ and $b_k$, the usual trigonometric coefficients, by defining $a_o \equiv \frac{1}{2\pi} \int_o^{2\pi} f_p(t)dt$, $h_p^*(s)$ can be written as:

(1) $$h_p^*(s) = \sum_{k=0}^{n} \frac{a_k s + kb_k}{s^2 + k^2} = \frac{P}{Q} = \frac{P_1}{2Q_1} + \frac{P_2}{2Q_2}$$

(2) $h^*(s) = \frac{P_1 P_2}{2P}$, therefore the poles of $h^*(s)$ are located at the zeros of $P(s)$. These zeros are $2n$ in number and are located in the left half plane (by assumption). No other general properties of these zeros are readily apparent, and it appears that the problem of locating them would be of

some degree of difficulty. This problem can be avoided, however, if lossless networks terminated in resistances are acceptable.

Recalling from Chapter I that $h_1^*(s)$ and $h_2^*(s)$ are respectively twice the sums of the odd order and even order terms of $h_p^*(s)$

$$(3) \qquad h_1^*(s) = 2 \sum_{k=0,\,2,\,4\cdots}^{n} \frac{a_k s + kb_k}{s^2 + k^2} = 2\left\{\frac{a_0}{s} + \frac{a_2 s + 2b_2}{s^2 + 4} + \frac{a_4 s + 4b_4}{s^2 + 16} + \cdots\right\} = \frac{P_1}{Q_1}.$$

$$(4) \qquad h_2^*(s) = 2 \sum_{k=1,3,5\cdots}^{n} \frac{a_k s + kb_k}{s^2 + k^2} = 2\left\{\frac{a_1 s + b_1}{s^2 + 1} + \frac{a_3 s + 3b_3}{s^2 + 9} + \frac{a_5 s + 5b_5}{s^2 + 25} \cdots\right\} = \frac{P_2}{Q_2}.$$

Note now that if odd $a_k$ and even $b_k$ are zero then $P_1$, $P_2$, and $Q_2$ are all even functions of s, while $Q_1$ is an odd function of s. Also if even $a_k$ and odd $b_k$ are zero then $P_1$, $Q_1$, and $Q_2$ are even in s, while $P_2$ is odd in s. Thus the product $P_1Q_2$ is always even in s and the product $P_2Q_1$ is always odd in s.

From equation (1) it is evident that

$$(5) \qquad P = \frac{1}{2}\left\{P_1Q_2 + P_2Q_1\right\},$$

so that for the special conditions that odd cosine and even sine coefficients be zero, or that even cosine and odd sine coefficients be zero, the odd and even s terms in $P(s)$ are identified with the poles and zeros of $h_1^*(s)$ and $h_2^*(s)$.

The necessary and sufficient conditions[3] for a polynomial with real coefficients to be a Hurwitz polynomial are that the zeros of its odd and even parts lie on the j axis where they mutually separate each other. Further, the ratio of the even to the odd parts of a Hurwitz polynomial is a reactance (susceptance) function. This implies, for P Hurwitz, that the zeros of $P_1$ and $P_2$ lie on the j axis and that the function $\frac{P_1Q_2}{P_2Q_1}$ be a reactance function, for the special conditions considered. The following manipulations are therefore suggested:

$$h^*(s) = \frac{P_1P_2}{2P} = \frac{P_1P_2}{P_1Q_2 + P_2Q_1} ,$$

dividing numerator and denominator by $P_2Q_1$ yields:

$$h^*(s) = \frac{\frac{P_1}{Q_1}}{1+ \frac{P_1Q_2}{P_2Q_1}}$$

Now associate h (s) with $Z_{12}(s) = \frac{z_{12}}{1+ z_{22}}$ which is the form for the transfer impedance of a lossless network terminated in a one ohm resistance. $z_{12}$ and $z_{22}$ are

respectively the transfer and driving point functions for the lossless part of the network.

In making this association the following conditions are noted:

(a) $z_{22}$ is a reactance function.

(b) $z_{12}$ has its poles and zeros restricted to the j axis.

(c) all the poles of $z_{12}$ are contained in $z_{22}$.

(d) $z_{22}$ has poles, at the zeros of $P_2$, which are not contained in $z_{12}$.

Conditions (a), (b), and (c) are sufficient[2.] to insure synthesis as a lossless ladder network terminated in a resistance. The poles of $z_{22}$ which are not contained in $z_{12}$ represent lossless parallel tuned circuits in series with the load resistor.

At this point the formal procedure of separating $h_1^*(s)$ into two parts, each of which fulfills one set of the requirements for this type of synthesis, would lead to a general synthesis procedure. It is more enlightening, however, to first observe the time form of the transients which these conditions represent. Using Fourier series coefficients and noting that $f_p(t) \equiv 0$ for the interval $\pi < t < 2\pi$

$$a_o \equiv \frac{1}{2\pi} \int_o^\pi f_p(t)dt$$

$$a_k \equiv \frac{1}{\pi} \int_o^\pi f_p(t)\cos kt\, dt$$

$$b_k \equiv \frac{1}{\pi} \int_o^\pi f_p(t)\sin kt\, dt$$

Since even cosine terms and odd sine terms have even symmetry about $\frac{\pi}{2}$, if $f_p(\frac{\pi}{2}+t) = f_p(\frac{\pi}{2}-t)$, then odd $a_k$ and even $b_k$ are zero. And since odd cosine terms and even sine terms have odd symmetry about $\frac{\pi}{2}$, if $f_p(\frac{\pi}{2}+t) = -f_p(\frac{\pi}{2}-t)$, then even $a_k$ and odd $b_k$ are zero. Thus the special conditions occur when the transient has even or odd symmetry about its midpoint.

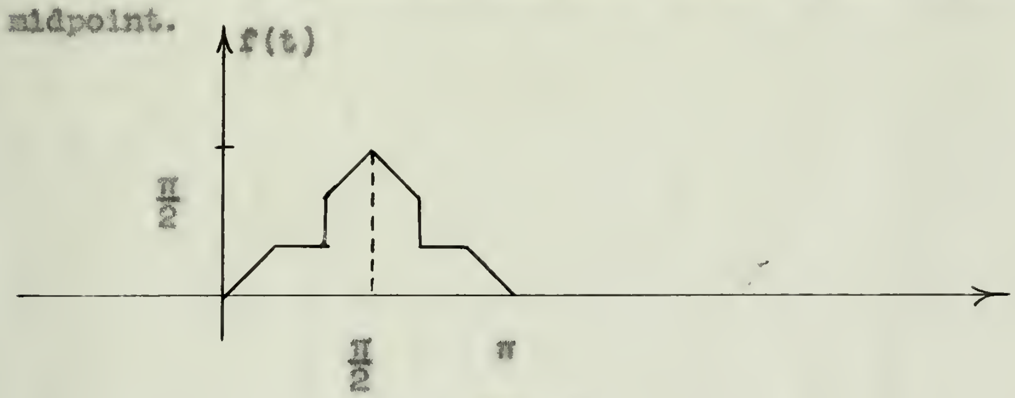



Fig. 4
Transient For Which Resulting Odd $a_k$ And Even $b_k$ Are Zero

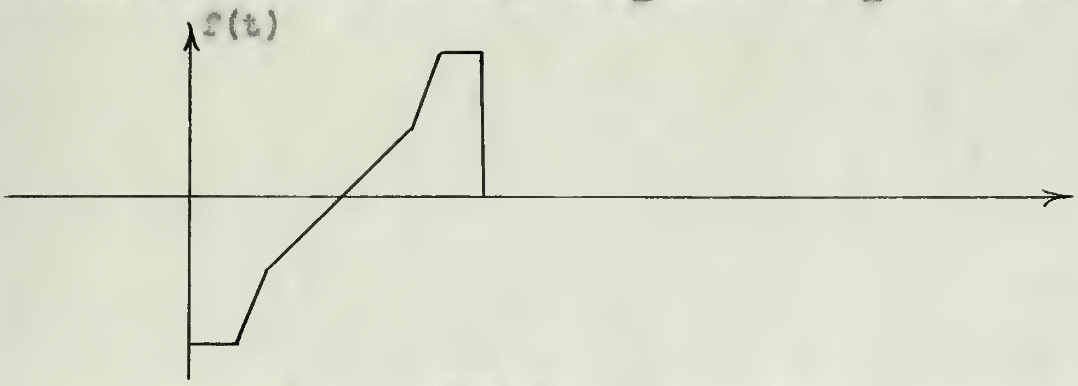



Fig. 5
Transient For Which Resulting Even $a_k$ And Odd $b_k$ Are Zero

Any arbitrary transient may be considered as the sum of two other transients, each having one of the above types of symmetry. To synthesize a network for an arbitrary transient response, then, first decompose the transient into its components having odd and even symmetry about the mid-point. Synthesize a network for each component. After suitable impedance leveling, connect the networks back to back and the overall transfer impedance function is realized.[2]

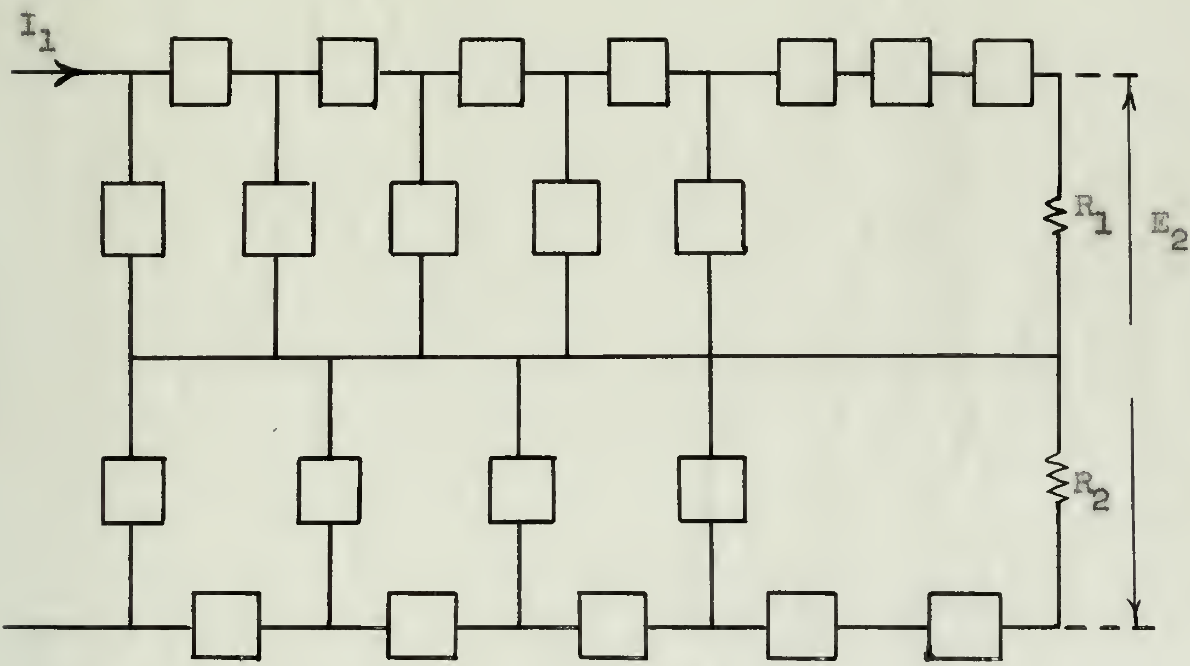
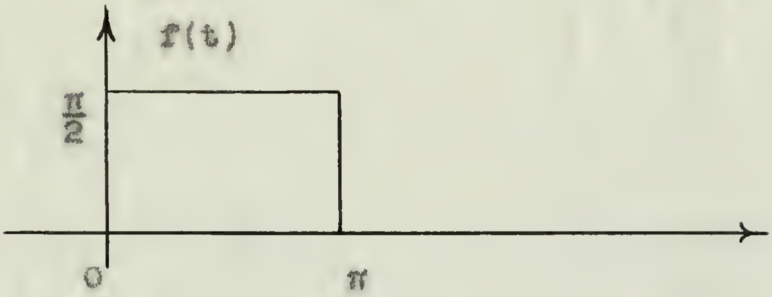



Fig. 6
A Form Of Network To Synthesize For A Realizable Transient Response

# CHAPTER III

## AN EXAMPLE OF THE SYNTHESIS PROCEDURE

Let the desired transient response to a unit impulse input be a rectangular one as shown:




Then $f_p(t)$ is


$$a_o = \frac{1}{2\pi} \int_o^{2\pi} f_p(t)dt = \frac{\pi}{4}$$

$$a_k = \frac{1}{\pi} \int_o^{2\pi} f_p(t)\cos kt\, dt = \frac{1}{\pi} \int_o^{\pi} \frac{\pi}{2} \sin kt\, dt = 0 \qquad (k \neq 0$$

$$b_k = \frac{1}{\pi} \int_o^{2\pi} f_p(t)\sin kt\, dt = \frac{1}{\pi} \int_o^{\pi} \frac{\pi}{2} \sin kt\, dt = \frac{1}{2k}(1-\cos k\pi$$

$$\therefore \quad b_k = \frac{1}{k} \text{ for } k \text{ odd}$$

$$b_k = 0 \text{ for } k \text{ even.}$$

Note that odd $a_k$ and even $b_k$ are zero since the transient has even symmetry about its midpoint. In this instance all $a_k$ are zero except $a_o$. This fact serves to simplify the network rather than invalidate the procedures developed.

Assume that it has been established that an approximation of six terms will be sufficiently accurate.

$$h_p^*(s) = \sum_{k=0}^{6} \frac{a_k s + kb_k}{s^2 + k^2} = \frac{\pi/4}{s} + \frac{1}{s^2+1} + \frac{1}{s^2+9} + \frac{1}{s^2+25}$$

$$h_1^*(s) = \frac{\pi/4}{s} = \frac{P_1}{Q_1}$$

$$h_2^*(s) = \frac{1}{s^2+1} + \frac{1}{s^2+9} + \frac{1}{s^2+25} = \frac{P_2}{Q_2}$$

$$P_1 = \frac{\pi}{4}$$

$$Q_1 = s$$

$$P_2 = (s^2+1)(s^2+9)+(s^2+1)(s^2+25)+(s^2+9)(s^2+25)$$
$$= 3(s^2+4.61132)(s^2+18.7702)$$

$$Q_2 = (s^2+1)(s^2+9)(s^2+25)$$

$$z_{12}(s) = \frac{P_1}{Q_1} = \frac{\pi/4}{s}$$

$$z_{22}(s) = \frac{P_1 Q_2}{P_2 Q_1} = \frac{(\pi/4)(s^2+1)(s^2+9)(s^2+25)}{3s\ (s^2+4.61132)(s^2+18.7702)}$$

Removing the factor of $\frac{\pi}{12}$ results in the following:

$$z_{22} = s + \frac{2.606}{s} + \frac{4.686s}{s^2+4.61132} + \frac{4.094s}{s^2+18.72202}$$

$$R = \frac{12}{\pi}$$

$$z_{12} = \frac{3}{s}$$

If only the rectangular form of the transient be required the impedance level may be reduced by a factor of $\frac{3}{2.606}$ yielding

$$z_{22} = s + \frac{2.606}{s} + \frac{4.686s}{s^2+4.61132} + \frac{4.094s}{s^2+18.72202}$$

$$R = 3.318$$

$$z_{12} = \frac{2.606}{s}$$

The network is then realized as

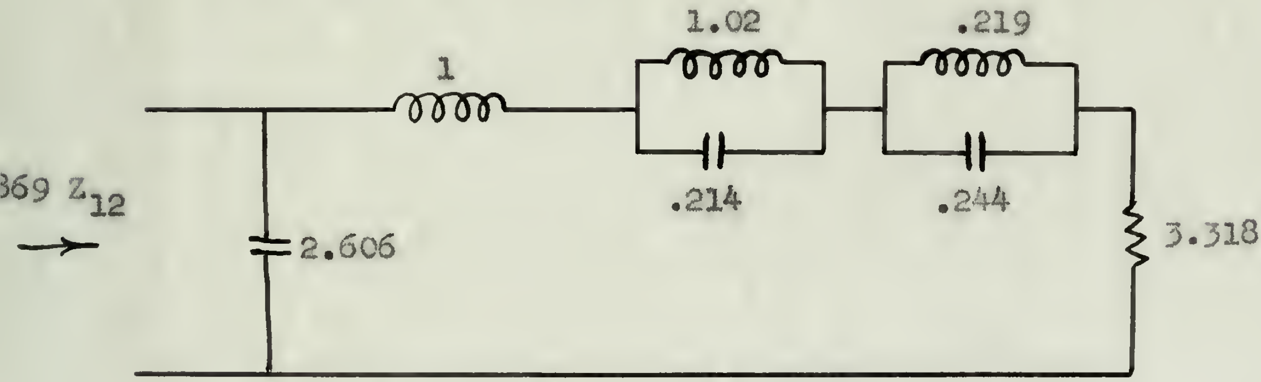



Fig. 8
Network For Approximating Rectangular Transient Response To A Unit Impulse
(Ohms, Henries, Farads)

Removing the factor of $\frac{3}{12}$ results in the following:

$$z_{22} = s + \frac{4.636}{s} + \frac{1.365s}{s^2+4.61132} + \frac{4.094s}{s^2+18.72202}$$

$$R = \frac{12}{?}$$

$$z_{12} = \frac{3}{s}$$

If only the rectangular form of the transient be required the impedance level may be reduced by a factor of $\frac{3}{2.606}$ yielding

$$z_{22} = s + \frac{2.606}{s} + \frac{4.035s}{s^2+4.61132} + \frac{4.094s}{s^2+18.72202}$$

$$R = 3.518$$

$$z_{12} = \frac{2.606}{s}$$

The network is then realized as

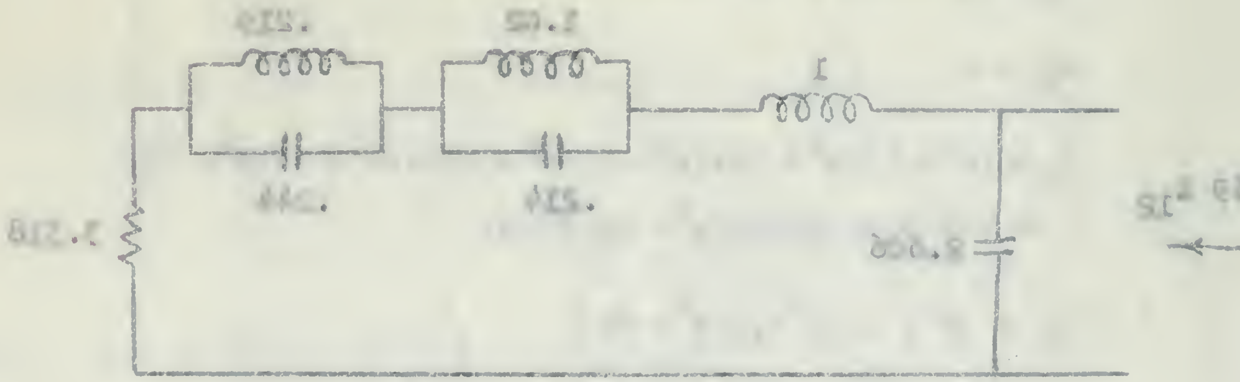



Fig. 6
Network For Approximating Rectangular Transient Response To A Unit Impulse
(Crum, Fournier, Parrish)

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Thesis
R67 Rodman 17228
Network synthesis for pre-
scribed transient response
using trigonometric series
approximations.